Math 150, *Lecture Notes- Bonds* **Name____________**

***Section 5.4** Exponential Functions: Differentiation and Integration*

The Natural Exponential Function

The function $f(x) = \ln x$ is increasing on its entire domain, and therefore it has an inverse function f^{-1}. The domain of f^{-1} is the set of all reals, and the range is the set of positive reals, as shown in Figure 5.19. So, for any real number x,

$$f(f^{-1}(x)) = \ln[f^{-1}(x)] = x. \qquad x \text{ is any real number.}$$

If x happens to be rational, then

$$\ln(e^x) = x \ln e = x(1) = x. \qquad x \text{ is a rational number.}$$

Because the natural logarithmic function is one-to-one, you can conclude that $f^{-1}(x)$ and e^x agree for *rational* values of x. The following definition extends the meaning of e^x to include *all* real values of x.

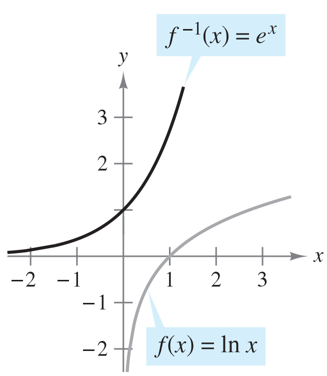
Definition of the Natural Exponential Function

The inverse function of the natural logarithmic function $f(x) = \ln x$ is called the **natural exponential function** and is denoted by

$$f^{-1}(x) = e^x.$$

That is,

$$y = e^x \quad \text{if and only if} \quad x = \ln y.$$



The inverse function of the natural logarithmic function is the natural exponential function.
Figure 5.19

The inverse relationship between the natural logarithmic function and the natural exponential function can be summarized as follows.

$$\ln(e^x) = x \quad \text{and} \quad e^{\ln x} = x \qquad \text{Inverse relationship}$$

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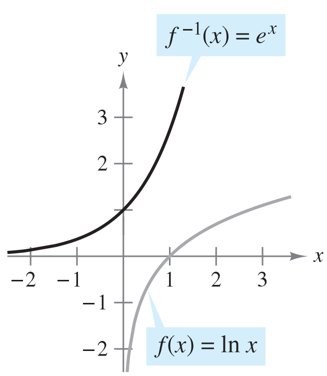

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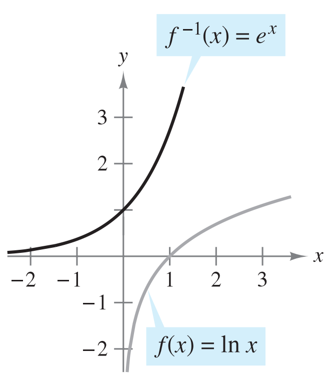

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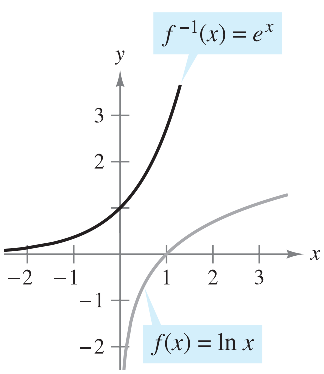

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Ex.1 Solving Exponential Equations

Solve $7 = e^{x+1}$.

$\ln(7) = \ln(e^{x+1})$

$\ln(7) = x+1$

$\ln(7) - 1 = x + 1 - 1$

$\ln(7) - 1 = x$

Ex.2 Solving a Logarithmic Equation

Solve $\ln(2x - 3) = 5$.

$e^{\ln(2x-3)} = e^5$

$2x - 3 = e^5$

$2x - 3 + 3 = 3 + e^5$

$2x = 3 + e^5$

$\frac{2x}{2} = \frac{3+e^5}{2}$

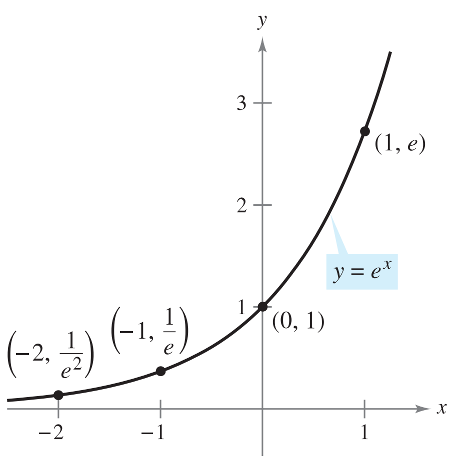
$x = \frac{3+e^5}{2}$

THEOREM 5.10 Operations with Exponential Functions

Let a and b be any real numbers.

1. $e^a e^b = e^{a+b}$
2. $\dfrac{e^a}{e^b} = e^{a-b}$

In Section 5.3, you learned that an inverse function f^{-1} shares many properties with f. So, the natural exponential function inherits the following properties from the natural logarithmic function (see Figure 5.20).



The natural exponential function is increasing, and its graph is concave upward.
Figure 5.20

Properties of the Natural Exponential Function

1. The domain of $f(x) = e^x$ is $(-\infty, \infty)$, and the range is $(0, \infty)$.
2. The function $f(x) = e^x$ is continuous, increasing, and one-to-one on its entire domain.
3. The graph of $f(x) = e^x$ is concave upward on its entire domain.
4. $\displaystyle\lim_{x\to-\infty} e^x = 0$ and $\displaystyle\lim_{x\to\infty} e^x = \infty$

Derivatives of Exponential Functions

One of the most intriguing (and useful) characteristics of the natural exponential function is that *it is its own derivative*. In other words, it is a solution to the differential equation $y' = y$. This result is stated in the next theorem.

THEOREM 5.11 Derivative of the Natural Exponential Function

Let u be a differentiable function of x.

1. $\frac{d}{dx}[e^x] = e^x$
2. $\frac{d}{dx}[e^u] = e^u\frac{du}{dx}$

PROOF

Let $y = e^x$, find $\frac{dy}{dx}$

$\ln(y) = \ln e^x,$

$\ln(y) = x \cdot \ln e,$

$\ln(y) = x \cdot 1$

$\frac{d}{dx}[\ln(y)] = \frac{d}{dx}(x)$

$\frac{1}{y} \cdot \frac{dy}{dx} = 1$

$y \cdot \frac{1}{y} \cdot \frac{dy}{dx} = 1 \cdot y$

$\frac{dy}{dx} = y$

$\frac{dy}{dx} = e^x$ ✓

Ex.3 Differentiating Exponential Functions

a. $\frac{d}{dx}[e^{2x-1}] = e^u\frac{du}{dx} = 2e^{2x-1}$ $\qquad u = 2x - 1$

b. $\frac{d}{dx}[e^{-3/x}] = e^u\frac{du}{dx} = \left(\frac{3}{x^2}\right)e^{-3/x} = \frac{3e^{-3/x}}{x^2}$ $\qquad u = -\frac{3}{x}$

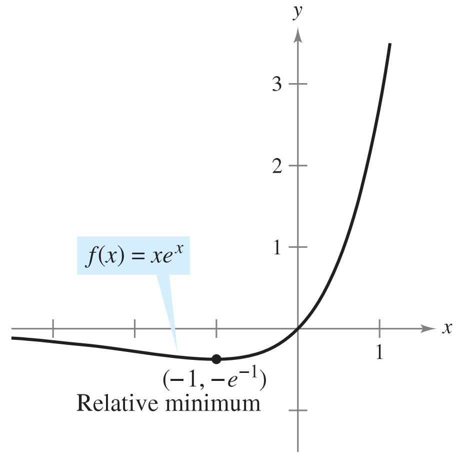
Let $u = \frac{-3}{x} = -3x^{-1}$

$\frac{du}{dx} = -3 \cdot [-1 \cdot x^{-2}]$

$\frac{du}{dx} = 3x^{-2} = \frac{3}{x^2}$

Ex.4 Locating Relative Extrema

Find the relative extrema of $f(x) = xe^x$.



The derivative of f changes from negative to positive at $x = -1$.
Figure 5.21

$$f'(x) = \frac{d}{dx}\left(xe^x\right)$$

$$f'(x) = x \cdot \frac{d}{dx}(e^x) + e^x \cdot \frac{d}{dx}(x)$$

$$f'(x) = x \cdot e^x + e^x \cdot 1$$

$$f'(x) = e^x(x+1)$$

$$f''(x) = \frac{d}{dx}\left[e^x(x+1)\right]$$

$$f''(x) = e^x \cdot \frac{d}{dx}[x+1] + (x+1) \cdot \frac{d}{dx}(e^x)$$

$$f''(x) = e^x \cdot 1 + (x+1) \cdot e^x$$

$$f''(x) = e^x[1 + (x+1)]$$

$$f''(x) = e^x(x+2)$$

critical numbers:

Ⓐ $f'(x) = 0$

$e^x(x+1) = 0$

$e^x = 0$, or $x+1 = 0$

never $\qquad x = -1$

Ⓑ $f'(x)$ is undefined

$e^x(x+1)$ is never undefined.

Test $x = -1$ in $f''(x) = e^x(x+2)$

$$f''(-1) = e^{(-1)}((-1)+2)$$

$$f''(-1) = \frac{1}{e} > 0$$

concave up

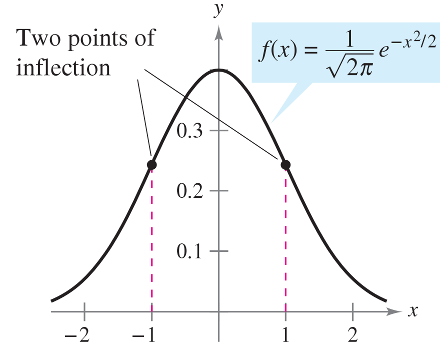
Relative Minimum

Ex.5 The Standard Normal Probability Density Function

Show that the *standard normal probability density function*

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

has points of inflection when $x = \pm 1$.



The bell-shaped curve given by a standard normal probability density function
Figure 5.22

$f'(x) = \frac{d}{dx}\left[\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\right]$

$f'(x) = \frac{1}{\sqrt{2\pi}} \cdot \frac{d}{dx}\left[e^{-\frac{x^2}{2}}\right]$

$f'(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \cdot \frac{d}{dx}\left[-\frac{x^2}{2}\right]$

chain Rule

$f'(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \cdot \left[-\frac{1}{2} \cdot 2x\right]$

$f'(x) = \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{x^2}{2}} \cdot [-x]$

$f'(x) = \frac{-1}{\sqrt{2\pi}} x e^{-\frac{x^2}{2}}$

$f''(x) = \frac{d}{dx}\left[\frac{-1}{\sqrt{2\pi}} \cdot x e^{-\frac{x^2}{2}}\right] = \frac{-1}{\sqrt{2\pi}} \cdot \frac{d}{dx}\left[x e^{-\frac{x^2}{2}}\right]$

$f''(x) = \frac{-1}{\sqrt{2\pi}} \cdot \left[x \cdot \frac{d}{dx}\left[e^{-\frac{x^2}{2}}\right] + e^{-\frac{x^2}{2}} \cdot \frac{d}{dx}(x)\right]$

$f''(x) = \frac{-1}{\sqrt{2\pi}} \left[x \cdot e^{-\frac{x^2}{2}} \cdot (-x) + e^{-\frac{x^2}{2}} \cdot 1\right]$

chain Rule

$f''(x) = \frac{-1}{\sqrt{2\pi}} \left[e^{-\frac{x^2}{2}} \cdot (-x^2 + 1)\right]$

Possible Points of Inflection:

(A) $f''(x) = 0$

$0 = -\frac{1}{\sqrt{2\pi}} \left[\frac{-x^2+1}{e^{\frac{x^2}{2}}}\right]$

$0 = -x^2 + 1$

$x^2 = 1$, $x = 1$ or $x = -1$

(B) $f''(x)$ is undefined

$-\frac{1}{\sqrt{2\pi}} \left[\frac{-x^2+1}{e^{\frac{x^2}{2}}}\right]$ is never undefined

NONE

Test $-2 = x$

$0 = x$

$2 = x$

Ex.6 Shares Traded

The numbers y of shares traded (in millions) on the New York Stock Exchange from 1990 through 2005 can be modeled by

$$y = 39{,}811e^{0.1715t}$$

where t represents the year, with $t = 0$ corresponding to 1990. At what rate was the number of shares traded changing in 2000? *(Source: New York Stock Exchange, Inc.)*

$$\frac{d}{dt}[y] = \frac{d}{dt}[39{,}811\, e^{0.1715t}]$$

$$\frac{dy}{dt} = 39{,}811 \cdot \frac{d}{dt}[e^{0.1715t}]$$

chain Rule

$$\frac{dy}{dt} = 39{,}811 \cdot e^{0.1715t} \cdot \frac{d}{dt}[0.1715t]$$

$$\frac{dy}{dt} = 39{,}811 \cdot e^{0.1715t} \cdot 0.1715$$

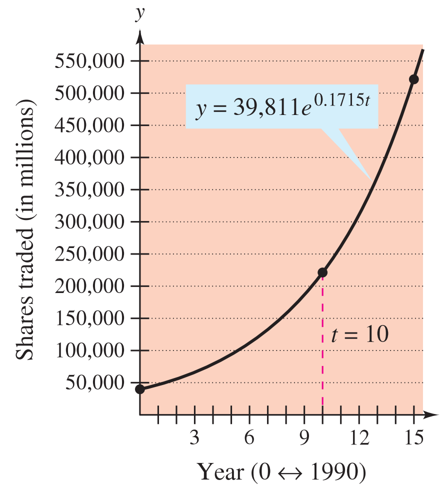


Figure 5.23

Integrals of Exponential Functions

Each differentiation formula in Theorem 5.11 has a corresponding integration formula.

THEOREM 5.12 Integration Rules for Exponential Functions

Let u be a differentiable function of x.

1. $\int e^x\,dx = e^x + C$ **2.** $\int e^u\,du = e^u + C$

Ex.7 Integrating Exponential Functions

Find $\int e^{3x+1}\,dx$.

$$\int e^{3x+1}\,dx = \int e^u \cdot \left(\frac{du}{3}\right)$$
$$= \frac{1}{3} \cdot \int e^u\,du$$
$$= \frac{1}{3} e^u + C$$
$$= \frac{1}{3} e^{3x+1} + C$$

Let $u = 3x+1$

$\frac{du}{dx} = 3$

$du = \frac{du}{dx} \cdot dx$

$du = 3 \cdot dx$

$\frac{du}{3} = \frac{3 \cdot dx}{3}$

$\frac{du}{3} = dx$

Ex.8 Integrating Exponential Functions

Find $\int 5xe^{-x^2}\,dx$.

$$\int 5xe^{-x^2}\,dx$$
$$= 5 \cdot \int xe^{-x^2}\,dx$$
$$= 5 \cdot \int e^u \cdot \left(\frac{du}{-2}\right)$$
$$= \frac{-5}{2} \cdot \int e^u\,du$$
$$= \frac{-5}{2} \cdot e^u + C = \frac{-5}{2} e^{-x^2} + C$$

Let $u = -x^2$

$\frac{du}{dx} = -2x$

$du = \frac{du}{dx} \cdot dx$

$\frac{du}{-2} = \frac{-2x \cdot dx}{-2}$

$\frac{du}{-2} = x \cdot dx$

Ex.9 Integrating Exponential Functions

a. Find $\int \frac{e^{1/x}}{x^2}\, dx = \int e^u \cdot (-du)$

$= -\int e^u\, du$

$= -e^u + C$

$= -e^{\frac{1}{x}} + C$

let $u = \frac{1}{x} = x^{-1}$

$\frac{du}{dx} = -1 \cdot x^{-2}$

$\frac{du}{dx} = \frac{-1}{x^2}$

$du = \frac{du}{dx} \cdot dx$

$du = \frac{-1}{x^2} \cdot dx$

$-1 \cdot du = -1 \cdot \left[-\frac{1}{x^2} dx\right]$

$-du = \frac{1}{x^2}\, dx$

b. Find $\int \sin x\, e^{\cos x}\, dx = \int e^u [-du]$ let $u = \cos(x)$

$\frac{du}{dx} = -\sin(x)$

$= -\int e^u\, du$

$du = \frac{du}{dx} \cdot dx$

$= -e^u + C$

$du = -\sin(x) \cdot dx$

$= -e^{\sin(x)} + C$

$-1 \cdot du = -1 \cdot [-\sin(x)\, dx]$

$-du = \sin(x)\, dx$

Ex.10 Finding Areas Bounded by Exponential Functions

Evaluate each definite integral.

a. $\int_0^1 e^{-x}\,dx = \int_{u=0}^{u=-1} e^{u}\cdot(-du)$

Let $u=-x$

$\frac{du}{dx}=-1$

$du=\frac{du}{dx}\cdot dx$

$du=-1\cdot dx$

$\boxed{-du=dx}$

If $x=0$
$u=-0$
$u=0$

If $x=1$
$u=-(1)$
$u=-1$

$= -\int_0^{-1} e^u\,du$

$= -\left[-\int_{-1}^{0} e^u\,du\right]$

$= \int_{-1}^{0} e^u\,du$

$= \left[e^u\right]_{-1}^{0} = e^0 - e^{-1} = 1-\frac{1}{e} = \frac{e}{e}-\frac{1}{e} = \frac{e-1}{e} \approx 0.632$

b. $\int_0^1 \frac{e^x}{1+e^x}\,dx$

$= \int_{u=2}^{u=1+e} \frac{1}{u}\cdot du$

$= \left[\ln|u|\right]_2^{1+e}$

$= \ln|1+e| - \ln|2|$

$= \ln(1+e) - \ln(2)$

$= \ln\left(\frac{1+e}{2}\right)$

$\ln\left(\frac{1+e}{2}\right) \approx 0.62$

Let $u=1+e^x$

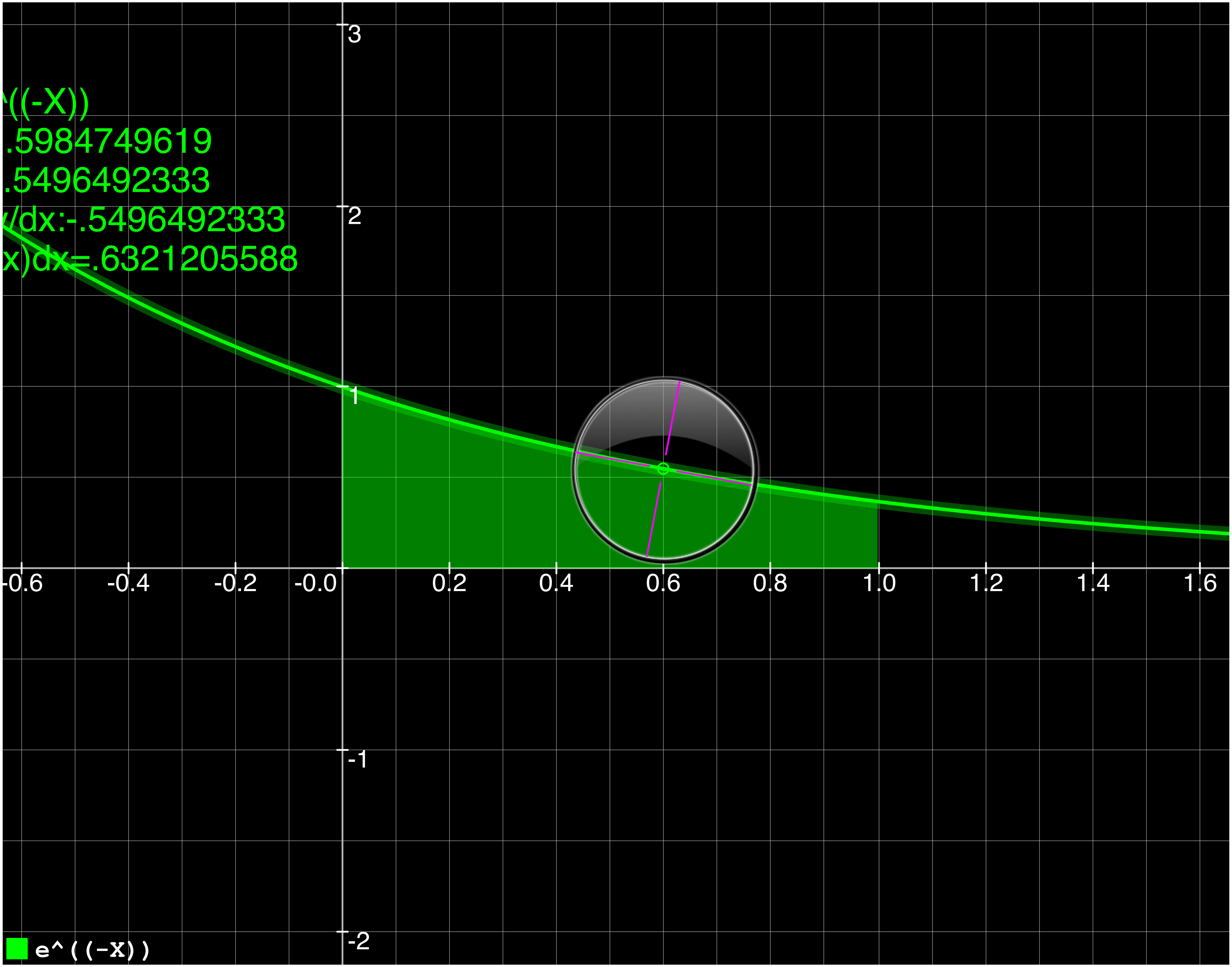
$\frac{du}{dx}=e^x$

$du=\frac{du}{dx}\cdot dx$

$du=e^x\cdot dx$

If $x=0$
$u=1+e^0$
$u=1+1$
$u=2$

If $x=1$
$u=1+e^1$
$u=1+e$


^((-X))
.5984749619
.5496492333
/dx:-.5496492333
x)dx=.6321205588
3
2
1
-1
-2
-0.6
-0.4
-0.2
-0.0
0.2
0.4
0.6
0.8
1.0
1.2
1.4
1.6
e^((-X))


3
2
1
-1
-2
-0.6
-0.4
-0.2
-0.0
0.2
0.4
0.6
0.8
1.0
1.2
1.4
1.6
e^((-X))

c. $\int_{-1}^{0} [e^x \cos(e^x)]\, dx$

$= \int_{u=\frac{1}{e}}^{u=1} \cos(u)\, du$

$= \left[\sin(u)\right]_{\frac{1}{e}}^{1}$

$= \sin(1) - \sin\left(\frac{1}{e}\right)$

$\sin(1) - \sin\left(\frac{1}{e}\right) \approx 0.4818$

Let $u = e^x$

$\frac{du}{dx} = e^x$

$du = \frac{du}{dx} \cdot dx$

$du = e^x \cdot dx$

If $x = -1$

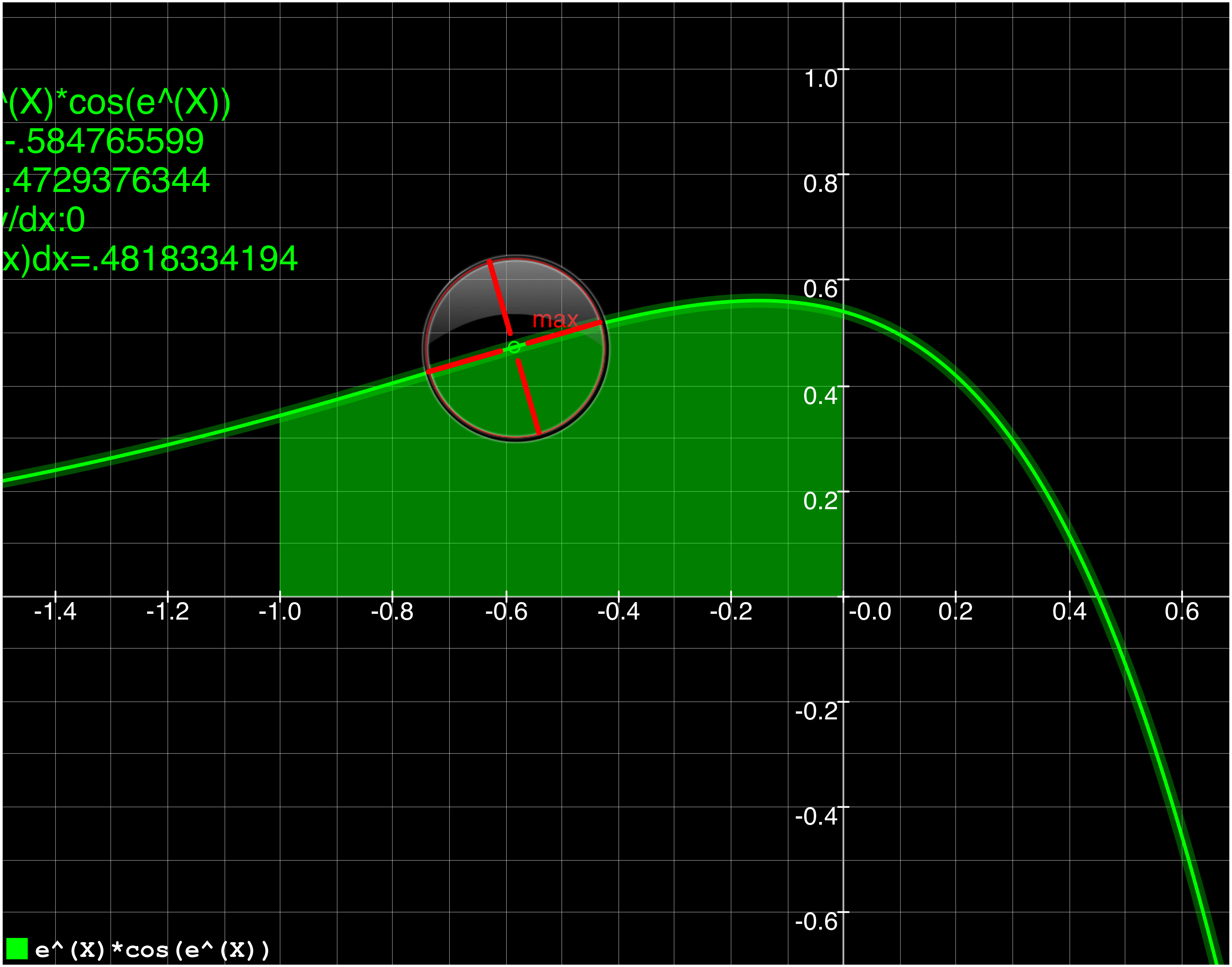
$u = e^{(-1)}$

$u = \frac{1}{e}$

If $x = 0$

$u = e^0$

$u = 1$


^(X)*cos(e^(X))
-.584765599
.4729376344
/dx:0
x)dx=.4818334194
max
1.0
0.8
0.6
0.4
0.2
-0.2
-0.4
-0.6
-1.4
-1.2
-1.0
-0.8
-0.6
-0.4
-0.2
-0.0
0.2
0.4
0.6
e^(X)*cos(e^(X))